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Homoenergetic oscillating solutions of the non-linear Boltzmann equation

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Abstract. The non-linear Boltzmann equation in the presence of external forces is considered. Exact inhomogeneous homoenergetic solutions are found by means of a non-isotropic generalisation of Nikolskii's transform. In particular, an interesting oscillating behaviour is obtained in the presence of an external time-independent force. For small values of the vorticity this oscillation becomes a sharp pulsation.

We study a gas of particles in d dimensions, in the presence of an external force per mass unit $\boldsymbol{a}(\boldsymbol{r}, t)$. The non-linear Boltzmann equation (NLBE) that governs the temporal evolution of the distribution function $f(\boldsymbol{r}, \boldsymbol{v}, t)$ is

$$\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\nabla_r+\boldsymbol{a}\cdot\nabla_v\right)f(\boldsymbol{r}, \boldsymbol{v}, t)=B[f,f] \tag{1}$$

where $B[f,f]$ is the bilinear term associated with elastic scattering collisions:

$$B[f,f]=\int \mathrm{d}\boldsymbol{v}_1\,\mathrm{d}\hat{\boldsymbol{n}}\,|\boldsymbol{v}-\boldsymbol{v}_1|\sigma\left(|\boldsymbol{v}-\boldsymbol{v}_1|, \frac{(\boldsymbol{v}-\boldsymbol{v}_1)\cdot\hat{\boldsymbol{n}}}{|\boldsymbol{v}-\boldsymbol{v}_1|}\right)\cdot[f(\boldsymbol{r}, \boldsymbol{v}', t)f(\boldsymbol{r}, \boldsymbol{v}_1', t) - f(\boldsymbol{r}, \boldsymbol{v}, t)f(\boldsymbol{r}, \boldsymbol{v}_1, t)]. \tag{2}$$

Here σ is the differential cross section for the intermolecular collision. The incoming and postcollisional velocities are related by

$$\boldsymbol{v}'=\tfrac{1}{2}(\boldsymbol{v}+\boldsymbol{v}_1)+\tfrac{1}{2}|\boldsymbol{v}-\boldsymbol{v}_1|\hat{\boldsymbol{n}} \tag{3}$$

$$\boldsymbol{v}_1'=\tfrac{1}{2}(\boldsymbol{v}+\boldsymbol{v}_1)-\tfrac{1}{2}|\boldsymbol{v}-\boldsymbol{v}_1|\hat{\boldsymbol{n}}. \tag{4}$$

In recent years, the discovery by Bobylev (1976) and Krook and Wu (1976) of an exact particular solution of the spatially uniform NLBE for Maxwell molecules has caused a revival of interest in this equation. Since then, significant progress has been achieved in the study of the space-independent Boltzmann equation (Ernst 1981, Bovylev 1984). In contrast, the quest for exact space-dependent solutions has not been so successful. Nikolskii (1964) discovered a transformation that makes it possible to construct space-dependent solutions of the NLBE without external forces for Maxwell molecules, starting from spatially uniform solutions. Recently Cornille (1986a, b)

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applied this transformation to a gas with power-law intermolecular forces. These solutions describe an expanding or collapsing gas, unless subject to particular time-dependent external forces. In the present paper we show that, by means of a non-isotropic generalisation of Nikolskii's transformation, it is possible to define explicit space-dependent solutions of the NLBE which remain confined in the presence of realistic time-independent external forces.

We look for a solution of the NLBE such that the spatial variable $\boldsymbol{r}$ appears in the distribution function only through the bulk velocity

$$\boldsymbol{u}(\boldsymbol{r}, t) = \frac{1}{\rho(\boldsymbol{r}, t)} \int f(\boldsymbol{r}, \boldsymbol{v}, t)\boldsymbol{v}\, \mathrm{d}\boldsymbol{v}. \tag{5}$$

Then we write $f(\boldsymbol{r}, \boldsymbol{v}, t) = f(\boldsymbol{c}(\boldsymbol{r}, \boldsymbol{v}, t), t)$, with $\boldsymbol{c}(\boldsymbol{r}, \boldsymbol{v}, t) = \boldsymbol{v} - \boldsymbol{u}(\boldsymbol{r}, t)$. It is easy to see that for this particular solution the moments formed with the velocity $\boldsymbol{c}$ may be functions of time but not of space coordinates; namely

density $$\rho = \int f(\boldsymbol{c}, t)\, \mathrm{d}\boldsymbol{v} \tag{6}$$

temperature $$T = \frac{1}{\rho(t) d} \int f(\boldsymbol{c}, t) \boldsymbol{c}^2\, \mathrm{d}\boldsymbol{v} \tag{7}$$

stress tensor $$M_{ij} = \int f(\boldsymbol{c}, t) c_i c_j\, \mathrm{d}\boldsymbol{v} \tag{8}$$

heat flux $$\boldsymbol{q} = \frac{1}{2} \int f(\boldsymbol{c}, t) \boldsymbol{c}^2 \boldsymbol{c}\, \mathrm{d}\boldsymbol{v} \tag{9}$$

are space independent. The corresponding balance equations of mass, momentum and energy are

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \nabla \cdot \boldsymbol{u} = 0 \tag{10}$$

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \boldsymbol{a} \tag{11}$$

$$\frac{\mathrm{d}T}{\mathrm{d}t} + \frac{2}{\rho d} \mathrm{Tr}(\mathbf{MA}) = 0 \tag{12}$$

with

$$A_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial r_j} + \frac{\partial u_j}{\partial r_i}\right) \tag{13}$$

being the symmetric rate-of-strain tensor. In order to make the previous balance equations compatible with each other, $\nabla \cdot \boldsymbol{u}$ in (10) must be space independent. Thus, we consider a flow for which the bulk velocity is an affine function of the position $\boldsymbol{u}(\boldsymbol{r}, t) = \boldsymbol{G}(t)\boldsymbol{r} + \boldsymbol{g}(t)$. Now, in order to solve the energy balance equation (12) in closed form, we suppose that only dilatation effects can occur, namely $A_{ij}(t) = A(t)\, \partial_{ij}$. Then the density and the temperature become

$$\rho(t) = \rho(0)\beta(t)^{-d} \tag{14}$$

$$T(t) = T(0)\beta(t)^{-2} \tag{15}$$

with

$$\beta(t)=\exp\int_0^t A(t')\,\mathrm{d}t'. \tag{16}$$

From the momentum balance equation (11), we see that the most general irrotational force compatible with this homoenergetic dilatation flow (Truesdell and Muncaster 1980), is

$$\boldsymbol{a}(\boldsymbol{r},t)=\frac{1}{\beta}\frac{\mathrm{d}^2\beta}{\mathrm{d}t^2}\boldsymbol{r}+\frac{1}{\beta^4}\mathbf{U}^2\boldsymbol{r}+\boldsymbol{b}(t) \tag{17}$$

where $\mathbf{U}$ a time-independent skew tensor. Now the momentum balance equation (11) can be explicitly solved:

$$\boldsymbol{u}(\boldsymbol{r},t)=\frac{1}{\beta}\frac{\mathrm{d}\beta}{\mathrm{d}t}\boldsymbol{r}+\frac{1}{\beta^2}\mathbf{U}\boldsymbol{r}+\frac{1}{\beta^2}\mathbf{Z}^{\mathrm{T}}\left[\boldsymbol{u}_0+\int_0^t\mathbf{Z}(t')\boldsymbol{b}(t')\,\mathrm{d}t'\right] \tag{18}$$

where the auxiliary matrix $\mathbf{Z}(t)$ is given by

$$\mathbf{Z}(t)=\beta(t)\exp\left(\mathbf{U}\int_0^t\beta(t')^{-2}\,\mathrm{d}t'\right). \tag{19}$$

Let us restrict this particular solution further by means of a generalised Nikolskii transformation $f(\boldsymbol{r},\boldsymbol{v},t)=F(\boldsymbol{q}(\boldsymbol{r},\boldsymbol{v},t),t)$ with $\boldsymbol{q}(\boldsymbol{r},\boldsymbol{v},t)=\mathbf{Z}(t)[\boldsymbol{v}-\boldsymbol{u}(\boldsymbol{r},t)]$. Considering an arbitrary inverse-power-law intermolecular potential $V\propto r^{-\mu}$ such that (Ernst 1981)

$$\sigma(v,\cos\theta)=v^{2(1-d)/\mu}\chi(\cos\theta) \tag{20}$$

the NBLE for this particular class of solution is

$$\frac{\mathrm{d}F}{\mathrm{d}t}=\beta^{(2-\mu)(d-1)/\mu}B[F,F]. \tag{21}$$

The auxiliary matrix $\mathbf{Z}(t)$ satisfies the orthogonality condition $\mathbf{Z}\mathbf{Z}^{\mathrm{T}}=\beta^2\mathbf{I}$. Therefore the change of variables $\boldsymbol{v}\to\boldsymbol{q}$ and $\hat{\boldsymbol{n}}\to\mathbf{Z}\hat{\boldsymbol{n}}/\beta$ renders the collision term unaltered, except for the β-dependent factor in (21). Finally, by redefining the time scale

$$\tau=\int_0^t\beta(t')^{(2-\mu)(d-1)/\mu}\,\mathrm{d}t' \tag{22}$$

we reduce the complete NLBE to the space-independent one without external forces

$$\frac{\mathrm{d}F}{\mathrm{d}\tau}=B[F,F]. \tag{23}$$

This equation represents one of the main results of the present paper, namely that a non-isotropic generalisation of the Nikolskii's transform makes it possible to build up a general homoenergetic dilating solution of the NLBE (1) from a space-independent one. Calculating the first moments of $\boldsymbol{q}$, we obtain the usual conservation laws of mass, momentum and energy for a homogeneous gas:

$$\int\mathrm{d}\boldsymbol{q}\,F(\boldsymbol{q},\tau)=\rho(0) \tag{24}$$

$$\int\mathrm{d}\boldsymbol{q}\,\boldsymbol{q}F(\boldsymbol{q},\tau)=0 \tag{25}$$

$$\int\mathrm{d}\boldsymbol{q}\,q^2F(\boldsymbol{q},\tau)=\rho(0)T(0)d. \tag{26}$$

Significant progress has been achieved in the study of the relaxation of the solution of the NLBE (23) to the equilibrium Maxwellian distribution

$$F_{\mathrm{M}}(q)=\frac{\rho(0)}{[2\pi T(0)]^{d/2}}\exp\left(-\frac{q^2}{2T(0)}\right). \tag{27}$$

Many existence and uniqueness problems have been solved for this relaxation process, and some exact analytical solutions are known. Actually, the explicit solution of the NLBE (23) is presently known for two kinds of interactions: the Maxwellian gas $\mu=2(d-1)$ (Ernst 1981), and the very hard particle (VHP) model (Hendriks and Ernst 1983). This VHP model is defined by the unphysical choice $\mu=-2(d-1)$. The scattering cross section (20) increases linearly with the relative velocity, whereas in real systems it is bounded by a constant.

In 1976 Bobylev used a Fourier transformation in velocity space and found a particular exact solution for Maxwell molecules. This solution—nowadays called the BKW solution—was simultaneously and independently discovered by Krook and Wu (1976) by means of a similarity technique. It is

$$F(q,\tau)=F_{\mathrm{M}}(q)(1-\nu)^{-d/2}\exp\left(-\frac{\nu\varepsilon}{1-\nu}\right)\left[1-\frac{\nu}{1-\nu}\left(\frac{d}{2}-\frac{\varepsilon}{1-\nu}\right)\right] \tag{28}$$

$$\nu(\tau)=\nu(0)\exp(-\lambda\tau) \qquad 0\leqslant\nu(0)\leqslant\frac{2}{d+2} \tag{29}$$

$$\lambda=\frac{\pi^{(d-1)/2}}{\Gamma(\frac{1}{2}d-1)}\int_0^\pi \chi(\cos\theta)\left(1-\cos^4\frac{\theta}{2}-\sin^4\frac{\theta}{2}\right)\mathrm{d}\theta \tag{30}$$

with $\varepsilon=q^2/2T(0)$ being the energy per thermal unit.

Furthermore, the general solution of (23) is known for this interaction model within a certain Hilbert space $L_2(R^d)$ with norm $\|F\|^2=\int|F(\boldsymbol{q},\tau)|^2/F_{\mathrm{M}}(q)\,\mathrm{d}\boldsymbol{q}$. This solution is given in the form of an expansion in terms of the eigenfunctions of the corresponding linearised collision integral. The time-dependent coefficients in this expansion satisfy a recursive solvable set of coupled non-linear equations. Convergence proofs of these series have been given for some classes of initial conditions (Hendriks and Nieuwenhuizen 1982). An extensive numerical study has also been performed for both spatially uniform (Ernst 1981) and non-uniform distributions (Barrachina and Garibotti 1986).

Example. For the particular case of a time-independent elastic force $\boldsymbol{a}(\boldsymbol{r},t)=-\omega^2\boldsymbol{r}+\boldsymbol{b}$, the tensor $\mathbf{U}^2$ in (17) must be proportional to the identity matrix, $\mathbf{U}^2=-\xi^2\mathbf{I}$ with ξ the initial vorticity of the flow (note that for odd dimension $\det(\mathbf{U})=0$ and ξ must be zero). Equation (17) can be easily solved. We obtain

$$\beta(t)=\sqrt{\beta_0(t)^2+(\xi/\omega)^2\sin^2(\omega t)} \tag{31}$$

with

$$\beta_0(t)=\cos(\omega t)+\frac{1}{\omega}\left.\frac{\mathrm{d}\beta}{\mathrm{d}t}\right|_{t=0}\sin(\omega t) \tag{32}$$

and the auxiliary $\mathbf{Z}(t)$ matrix is

$$\mathbf{Z}(t)=\beta_0(t)\mathbf{I}+\frac{1}{\omega}\sin(\omega t)\mathbf{U}. \tag{33}$$

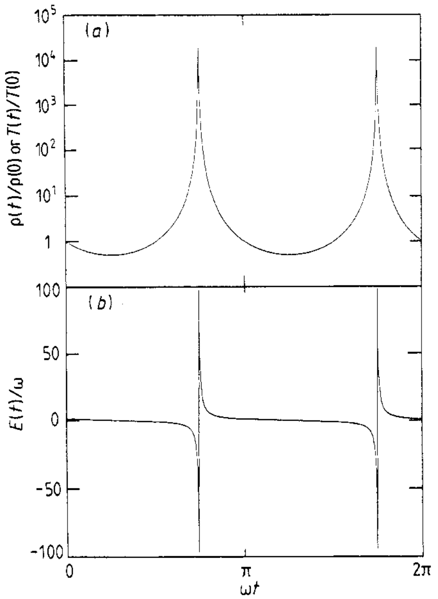

Figure 1. Time evolution of (a) the density, the temperature and (b) the stretching, for a two-dimensional homoenergetic dilatation flow in the presence of an external elastic force $\boldsymbol{a} = -\omega^2\boldsymbol{r}+\boldsymbol{b}$. The initial vorticity and stretching are $\xi = 0{,}01\omega$ and

$$\left.\frac{\mathrm{d}\beta}{\mathrm{d}t}\right|_{t=0} = \omega$$

respectively.

Thus the homoenergetic dilatation flow in the presence of the elastic force $\boldsymbol{a}(\boldsymbol{r}, t) = -\omega^2\boldsymbol{r}+\boldsymbol{b}$ has been completely solved. The time evolution of the density and the temperature of a two-dimensional gas is shown in figure 1(a). The mean stretching

$$E(t) = \frac{1}{d}\nabla\cdot\boldsymbol{u} = \frac{1}{\beta}\frac{\mathrm{d}\beta}{\mathrm{d}t} \tag{34}$$

is displayed in figure 1(b). An oscillating behaviour, which becomes a sharp pulsation for certain characteristic initial conditions, is obtained. In the absence of vorticity in the initial state, i.e. $\mathbf{U} = 0$, the solution diverges in a finite time. Actually, for irrotational initial conditions (Nikolskii 1964, Cornille 1985, 1986), purely repulsive or attractive time-independent elastic forces cause the gas to expand or collapse respectively. Cornille (1985) studied a non-rotating gas on a time-dependent elastic force. By coupling two circular functions, which made the force alternatively attractive and repulsive, he found an oscillating behaviour of the gas. Here we have shown that a similar oscillating behaviour is obtained with a simpler time-independent elastic force when the initial condition is less restrictive than that used by Cornille (1985).

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